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General form of coupling leading to synchronization of oscillating dynamical systems

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There are numerous examples in nature where oscillating or pulsing physical and biological systems automatically synchronize. Here we pose the question: Is there a general type of coupling that leads to such synchronization? It is demonstrated on numerous examples that dissipative coupling results in synchronization for a variety of systems. [S1063-651X(99)13308-7]

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INTRODUCTION

The old problem of synchronization of coupled oscillators has recently received renewed attention [1–5]. Physical systems such as Huygens pendulum clocks, hanging side by side self-synchronize, and so do a variety of biological systems, such as thousands of fireflies flashing in synchrony, pacemaker cells of the heart, etc. More recently synchronization of Josephson junction arrays has received much attention [6–9].

Much recent work has been carried out by Strogatz and co-workers [2–4], who study a system of coupled limit cycle oscillators described by

$$\dot{z}_j = z_j(1-|z_j|^2+i\omega)+\frac{K}{N}\sum_i (z_i-z_j),$$

where z_j is the position of the jth oscillator in the complex plane, it is clear that the uncoupled oscillators $(K=0)$ are attracted to the unit circle limit cycle and the motion of the individual oscillators on the unit circle is described by

$$\dot{\theta}_j=\omega+\frac{K}{N}\sum_i \sin(\theta_i-\theta_j).$$

Depending on the coupling constant K and the distribution of oscillators in the frequency domain synchronization can occur.

Here we are looking at a different aspect of synchronization. Since there are many different types of systems in nature that exhibit self-synchronization, it is reasonable to assume that there exists a general kind of coupling producing synchronization in a large variety of different systems. Typically one expects by increasing the number of dimensions of phase space, as one gets by coupling individual systems, to lead to more chaotic behavior. In these systems the opposite effect, synchronization leading to order is apparent.

Consider N identical oscillators, characterized by the equations

$$\dot{x}_i = f(x_i\, y_i), \qquad \dot{y}_i = g(x_i, y_i). \tag{1}$$

What should be the form of coupling added to these equations that leads to synchronization? It will be shown that a simple type of dissipative coupling leads to synchronization in a variety of systems.

COUPLING OF SIMPLE HAMILTONIAN OSCILLATORS

The simplest case to consider is the one with two harmonic oscillators with $f=y_i$, $g=-x_i$. Introduce dissipative coupling to get

$$\begin{aligned} \dot{x}_1 &= y_1 + C(x_2-x_1),\\ \dot{y}_1 &= -x_1,\\ \dot{x}_2 &= y_2 + C(x_1-x_2),\\ \dot{y}_2 &= -x_2, \end{aligned} \tag{2}$$

where C is the coupling constant, or in a different form

$$\begin{aligned} \ddot{x}_1 + C(\dot{x}_1-\dot{x}_2)+x_1 &= 0,\\ \ddot{x}_2 + C(\dot{x}_2-\dot{x}_1)+x_2 &= 0. \end{aligned} \tag{3}$$

Normal mode analysis leads to the frequencies $\omega=\pm 1$, corresponding to undamped in phase oscillators with $x_1 = x_2$, and $\omega = iC \pm (1-C^2)^{1/2}$, for the out of phase damped modes. Consequently arbitrary initial conditions lead to in phase oscillations, while the out of phase component is damped out. For N oscillators

$$\begin{aligned} \dot{x}_i &= y_i + C\left(D\sum_{k\neq i} x_k - x_i\right),\\ \dot{y}_i &= -x_i, \end{aligned} \tag{4}$$

with D a constant to be determined. One can use again normal mode analysis, but it is useful to introduce two different methods. In the first we consider the velocity difference vector field $(\dot{x}_i-\dot{x}_j, \dot{y}_i-\dot{y}_j)$ in the plane of position differences (x_i-x_j, y_i-y_j). Calculate now the divergence of this vector field

$$d(\dot{x}_i-\dot{x}_j)/d(x_i-x_j)+d(\dot{y}_i-\dot{y}_j)/d(y_i-y_j) = -C(1+D). \tag{5}$$

Since this quantity is negative, and the velocity difference vector field is zero for $x_i = x_j$, $y_i = y_j$, the velocity difference contracts to the point where all oscillators move together. This, however, does not necessarily lead to synchronized oscillations. When all x_i's are identical, the coupling term becomes $[D(N-1)-1]x_i$. Consequently when $D < (N-1)^{-1}$ the entire set of oscillators damps out while for $D > (N-1)^{-1}$, the oscillations amplitudes grow exponentially with time. So for synchronous oscillations of constant amplitude $D_c = (N-1)^{-1}$ should be chosen.

This great sensitivity to the coupling constant is characteristic of time-independent Hamiltonian systems. For $D < D_c$ energy is dissipated from the systems, while if $D > D_c$ energy is added. For driven Hamiltonian systems, or more general nonlinear oscillators this restriction does not apply as will be demonstrated in several examples.

An alternative method involves coupled Hamiltonian systems, with the equations of motion

$$\begin{aligned} \dot{x}_i &= \partial H_i/\partial y_i + c f_i(x_i, x_k), \\ \dot{y}_i &= -\partial H_i/\partial x_i, \end{aligned} \tag{6}$$

where H_i is the Hamiltonian of the ith oscillator. Define

$$F = \sum_i H_i \tag{7}$$

and calculate the time derivative

$$\begin{aligned} \dot{F} &= \sum_i (\partial H_i/\partial x_i \cdot \dot{x}_i + \partial H_i/\partial y_i \cdot \dot{y}_i) \\ &= \sum_i (\partial H_i/\partial x_i \cdot \partial H_i/\partial y_i + c\,\partial H_i/\partial x_i f_i \\ &\quad - \partial H_i/\partial y_i \cdot \partial H_i/\partial x_i) \\ &= c \sum_i \partial H_i/\partial x_i \cdot f_i . \end{aligned} \tag{8}$$

For coupled harmonic oscillators with $f_i = \Sigma_{k \neq i} x_k/(N-1) - x_i$, and $\partial H_i/\partial x_i = x_i$, one gets

$$\begin{aligned} \dot{F} &= c \sum_i x_i \left(\sum_{k \neq i} x_k/(N-1) - x_i \right) \\ &= -c \sum_i \left[x_i^2 - (N-1)^{-1} \sum_{k \neq i} x_i x_k \right] \\ &= -c/2(N-1) \sum_{i,k} (x_i - x_k)^2 . \end{aligned} \tag{9}$$

Therefore, F keeps decreasing until $x_i = x_k$ for all i and k, leading to synchronization.

Consider now oscillators governed by the Hamiltonian $H = y^2/2 + V(x)$, with the potential energy V having one or several minimal (such as $V = \cos x$), about which there is oscillatory motion. The equations of motion are

$$\begin{aligned} \dot{x}_i &= y_i + C \left[\sum_{k \neq i} x_k (N-1)^{-1} - x_i \right], \\ \dot{y}_i &= -\partial V/\partial x_i . \end{aligned} \tag{10}$$

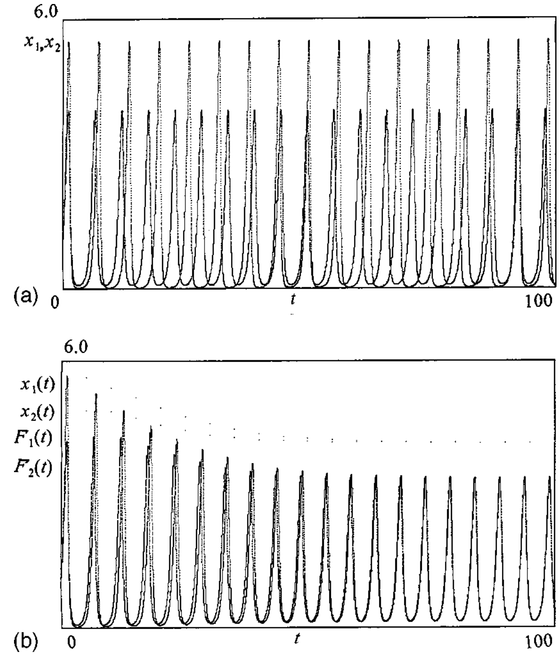

FIG. 1. Two Volterra oscillators $A=2$, $B=1$, with initial conditions $x_1 = x_2 = 1$ and $y_1 = 0.2$, $y_2 = 0.1$. (a) The oscillators are uncoupled. (b) Coupled with $C = 0.06$. The separate sets of points are F_1 and F_2 at different times.

Calculating now the divergence of the velocity difference (or relative velocity) field the result is again the one found in Eq. (5) for the harmonic oscillator. For $D = (N-1)^{-1}$ synchronization with constant amplitude oscillations is approached.

It should be noted that only the velocities $(\dot{x}_i, \dot{y}_i)$ converge not the positions. For a periodic potential, such as $\cos x$ different oscillators may be trapped in separate potential wells, while the velocities synchronize. In this case the coupling term in Eq. (10) approaches a nonzero constant asymptotically.

At this point it is useful to define what we mean by dissipative coupling. It is the kind of coupling where relative velocity between oscillators leads to phase space contraction, where the left hand side of Eq. (5) as applied to the coupling term is negative. All examples treated fall into this category. Just as in the previous examples linear coupling will be used.

SYNCHRONIZATION OF MORE GENERAL OSCILLATORS

Since many biological systems (e.g., fireflies) resemble self-synchronized pulsed systems, it is useful to consider the Volterra equations

$$\begin{aligned} \dot{x} &= Ax(1-y), \\ \dot{y} &= -By(1-x), \end{aligned} \tag{11}$$

which generate pulsed oscillations. Figure 1(a) shows computed solutions of these equations with $A=2$, $B=1$ and two initial conditions $x_1 = x_2 = 1$ and $y_1 = 0.2$, $y_2 = 0.1$.

Now one adds dissipative coupling to these equations in the form

$$\dot{x}_i = A x_i (1 - y_i) + C \sum_{k \neq i} x_k,$$
$$\dot{y}_i = -B y_i (1 - x_i). \tag{12}$$

With the substitution $x = e^{\zeta}$, $y = e^{\eta}$, Eq. (11) can be derived from the Hamiltonian

$$H = A[\exp(\eta) - \eta] + B[\exp(\zeta) - \zeta], \tag{13}$$

where η is the coordinate and ζ the momentum. It is useful to rewrite Eq. (12) in the new coordinates

$$\dot{\zeta}_i = A[1 - \exp(\eta_i)] + C \exp(-\zeta_i) \sum_{k \neq i} \exp(\zeta_k),$$
$$\dot{\eta}_i = -B[1 - \exp(\zeta_i)]. \tag{14}$$

For two coupled systems the ζ component of the relative velocity is

$$\dot{\zeta}_1 - \dot{\zeta}_2 = \varphi(\eta_1, \eta_2) + C[\exp(\zeta_2 - \zeta_1) - \exp(\zeta_1 - \zeta_2)] \tag{15}$$

and the divergence is,

$$\partial(\dot{\zeta}_1 - \dot{\zeta}_2)/\partial(\zeta_1 - \zeta_2) = -C[\exp(\zeta_2 - \zeta_1) + \exp(\zeta_1 - \zeta_2)], \tag{16}$$

a negative quantity. Since the relative velocity vanishes when $\zeta_1 = \zeta_2$, $\eta_1 = \eta_2$ the system converges to the synchronous state. Figure 1(b) shows this convergence, computed with the coupling constant $C = 0.06$.

For many coupled systems a modified version of Eq. (7) can be used

$$F = \sum_i [H_i - C(N-1)\eta_i] \tag{17}$$

with the time derivative

$$\begin{aligned} \dot{F} &= C \sum_i \partial H/\partial \zeta_i \left[\exp(-\zeta_i) \sum_{k \neq i} \exp(\zeta_k) - (N-1) \right] \\ &= BC \sum_i [\exp(\zeta_i) - 1] \left[\sum_{k \neq i} \exp(\zeta_k - \zeta_i) - (N-1) \right] \\ &= -2BC \sum_{i,k} \sinh^2[(\zeta_i - \zeta_k)/2]. \end{aligned} \tag{18}$$

This is clearly a negative quantity, converging to zero when $\zeta_i = \zeta_k$ for all i and k. The two sets of converging points in Fig. 1(b) are F_1 and F_2 at different times.

It should be noted that the coupling used in Eq. (12) has a different form from those used in earlier examples. While convergence has been proven, there is still the question of the asymptotic behavior of the system. Replacing Σx_k in Eq. (12) by $(N-1)x_i$, it is easy to see that neither $x = y = 0$ nor $x \to \infty$, $y \to \infty$ is a solution. In fact this leads to a minor modification of the Volterra equations, where in the first of Eq. (11), 1 is replaced by $1 + C(N-1)/A$.

Another example for synchronization by dissipative coupling is that of Van der Pol oscillators of the form [10]

$$\dot{x}_i = y_i + C \sum_{k \neq i} x_k,$$
$$\dot{y}_i = A(1 - x_i^2) y_i - x_i. \tag{19}$$

Individual Van der Pol oscillators $(C = 0)$, are harmonic oscillators with nonlinear damping, producing growing waves for $|x| < 1$, and damped waves for $|x| > 1$, so the wave settles for arbitrary initial conditions into self sustained oscillations. The damping coefficient $A > 0$, and small $(A \ll 1)$. For small A the wave form is nearly sinusoidal. In the following $C \ll 1$ will also be assumed.

For nearly sinusoidal waves one substitutes

$$x_i = R_i(t) \sin(t + \psi_i), \tag{20}$$

where R_i and ψ_i are slow variables. With this approximation (19) becomes

$$\begin{aligned} &2\dot{R}_i \cos(t + \psi_i) - 2R_i \dot{\psi}_i \sin(t + \psi_i) \\ &\quad - A[1 - R_i^2 \sin^2(t + \psi_i)] R_i \cos(t + \psi_i) \\ &\quad - C \sum_{k \neq i} R_k \cos(t + \psi_k) = 0. \end{aligned} \tag{21}$$

After multiplying by $\sin(t + \psi_i)$ and averaging over a period one gets

$$2R_i \dot{\psi}_i + C \sum_k R_k \sin(\psi_i - \psi_k) = 0. \tag{22}$$

For two coupled oscillations with $\psi_1 - \psi_2 = \Delta\psi$.

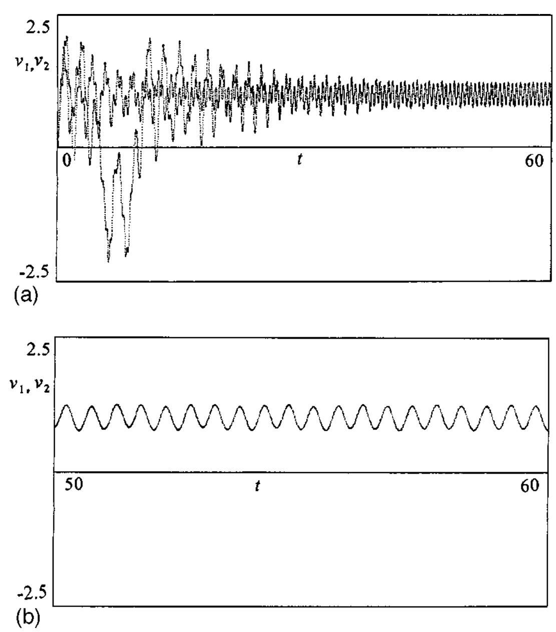

FIG. 2. Time development of two coupled driven systems $x_1(t)$ and $x_2(t)$. Equation (25), with $A = 5$, $C_1 = 0.05$, $C_2 = 0.2$, and initial conditions $y_1 = 0.1$, $y_2 = 0.2$, and $x_1 = x_2 = 0$. (a) $0 < t < 60$, (b) $50 < t < 60$.

$$\Delta\dot{\psi} = -(C/2)(R_1/R_2 + R_2/R_1)\sin(\Delta\psi). \tag{23}$$

For $C>0$, $\Delta\psi\to 0$ the oscillations have become synchronized, while for $C<0$, $\Delta\psi\to\pi$. For N oscillators for $C<0$, splay states are approached with $\Delta\psi\to 2\pi/N$. Multiplying Eq. (21) with $\cos(t+\psi_i)$ and averaging gives for two coupled oscillators

$$2\Delta\dot{R} = [A - C\cos\Delta\psi - (A/4)(R_1^2 + R_1R_2 + R_2^2)]\Delta R. \tag{24}$$

In the absence of coupling $R_1 = R_2 = 2$, so the right side is negative and $\Delta R\to 0$.

Numerical computation shows that synchronization occurs even when the fundamental frequencies of the individual oscillators are slightly different. [In the second part of Eq. (19), x_i is replaced with $(1+\varepsilon_i)x_i$.]

Finally, dissipatively coupled time dependent Hamiltonian systems are studied. Take, e.g., the equations

$$\dot{x}_i = A\sin(2\pi y_i)\sin(2\pi t) + C_1\sum_{k\neq i} x_k - C_2 x_i$$
$$\dot{y}_i = x_i, \tag{25}$$

where the C_1 term provides coupling and $C_2>0$ damping. The undamped uncoupled system $C_1 = C_2 = 0$ can exhibit regular or chaotic behavior, depending on initial conditions. Figure 2 shows the time development of two coupled systems described by Eq. (25) with $A=5$, $C_1 = 0.05$ and $C_2 = 0.2$. The initial conditions are chosen such that in the absence of damping and coupling the trajectories are chaotic. In Fig. 2 due to damping (C_2) the trajectories become regular (dissipation forces the trajectories to spiral into island chains) while C_1 produces synchronization.

It is easier to study two coupled standard maps with dissipation

$$P_i' = P_i + (K/2\pi)\sin(2\pi\cdot X_i) + C_1P_k - C_2P_i,$$
$$X_i' = X_i + P_i', \tag{26}$$

where for $i=1$, $k=2$ and $i=2$, $k=1$. The Jacobian determinant of the four by four matrix $D=(1-C_2)^{2-}C_1^2$, is less than one if $C_2<2$ and phase space contracts. One finds computationally a behavior quite similar to the one for Eq. (25), trapping in island chains for $C_2>0$ and synchronization (the two systems end up on the same island chain) when C_1 exceeds some critical value which depends on the initial conditions as well as the parameters K and C_2.

To summarize, a variety of different oscillator systems with dissipative coupling have been investigated. In each example the systems tended to synchronization.

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